DP Group 5:Mathematics HL

DP - Age 16-18

Aims

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- 1. enjoy mathematics, and develop an appreciation of the elegance and power of mathematics
- 2. develop an understanding of the principles and nature of mathematics
- 3. communicate clearly and confidently in a variety of contexts
- 4. develop logical, critical and creative thinking, and patience and persistence in problem-solving
- 5. employ and refine their powers of abstraction and generalization
- 6. apply and transfer skills to alternative situations, to other areas of knowledge and to future developments
- 7. appreciate how developments in technology and mathematics have influenced each other
- 8. appreciate the moral, social and ethical implications arising from the work of mathematicians and the applications of mathematics
- 9. appreciate the international dimension in mathematics through an awareness of the universality of mathematics and its multicultural and historical perspectives
- 10. appreciate the contribution of mathematics to other disciplines, and as a particular "area of knowledge" in the TOK course.

Objectives

Problem-solving is central to learning mathematics and involves the acquisition of mathematical skills and concepts in a wide range of situations, including non-routine, open-ended and real-world problems. Having followed a DP mathematics HL course, students will be expected to demonstrate the following.

- 1. Knowledge and understanding: recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of familiar and unfamiliar contexts.
- 2. Problem-solving: recall, select and use their knowledge of mathematical skills, results and models in both real and abstract contexts to solve problems.
- 3. Communication and interpretation: transform common realistic contexts into mathematics; comment on the context; sketch or draw mathematical diagrams, graphs or constructions both on paper and using technology; record methods, solutions and conclusions using standardized notation.
- 4. Technology: use technology, accurately, appropriately and efficiently both to explore new ideas and to solve problems.
- 5. Reasoning: construct mathematical arguments through use of precise statements, logical deduction and inference, and by the manipulation of mathematical expressions.
- 6. Inquiry approaches: investigate unfamiliar situations, both abstract and real-world, involving organizing and analysing information, making conjectures, drawing conclusions and testing their validity.

Syllabus - Topic 1 - Core: Algebra (30 hours)

The aim of this topic is to introduce students to some basic algebraic concepts and applications.

- 1.1 Arithmetic sequences and series; sum of finite arithmetic series; geometric sequences and series; sum of finite and infinite geometric series.
- Sigma notation.
- Applications.
- 1.2 Exponents and logarithms.
- Laws of exponents; laws of logarithms.
- Change of base.
- 1.3 Counting principles, including permutations and combinations.
- The binomial theorem: expansion of $(a+b)^n$, $n \in \mathbb{N}$.
- Not required:
- Permutations where some objects are identical.
- Circular arrangements.
- Proof of binomial theorem.
- 1.4 Proof by mathematical induction.
- 1.5 Complex numbers: the number
- $i=\sqrt{-1}$; the terms real part, imaginary part, conjugate, modulus and argument.
- Cartesian form $z=a+ib$.
- Sums, products and quotients of complex numbers.
- 1.6 Modulus–argument (polar) form $z=r(\cos\Theta+i\sin\Theta) = r \operatorname{cis} \Theta = re^{i\Theta}$.
- The complex plane.
- 1.7 Powers of complex numbers: de Moivre's theorem.
- nth roots of a complex number.
- 1.8 Conjugate roots of polynomial equations with real coefficients.
- 1.9 Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases where there is a unique solution, an infinity of solutions or no solution.

Syllabus - Topic 2 - Core: Functions and equations (22 hours)

The aims of this topic are to explore the notion of function as a unifying theme in mathematics, and to apply functional methods to a variety of mathematical situations. It is expected that extensive use will be made of technology in both the development and the application of this topic.

- 2.1 Concept of function $f : x \mapsto f(x)$ domain, range; image (value).
- Odd and even functions.
- Composite functions $f \circ g$.
- Identity function.
- One-to-one and many-to-one functions.
- Inverse function f^{-1}, including domain restriction. Self-inverse functions.

- 2.2 The graph of a function; its equation $y=f(x)$.
- Investigation of key features of graphs, such as maximum and minimum values, intercepts, horizontal and vertical asymptotes and symmetry, and consideration of domain and range.
- The graphs of the functions $y=|f(x)|$ and $y=f(|x|)$.
- The graph of $y=1/(f(x))$ given the graph of $y=f(x)$.
- 2.3 Transformations of graphs: translations; stretches; reflections in the axes.
- The graph of the inverse function as a reflection in $y=x$
- 2.4 The rational function $x \mapsto ((ax+b)/(cx+d))$, and its graph.
- The function $x \mapsto a^x$, $a > 0$, and its graph.
- The function $x \mapsto \log_a x, x>0$, and its graph.
- 2.5 Polynomial functions and their graphs.
- The factor and remainder theorems.
- The fundamental theorem of algebra.
- 2.6 Solving quadratic equations using the quadratic formula.
- Use of the discriminant $\Delta=b^2-4ac$ to determine the nature of the roots.
- Solving polynomial equations both graphically and algebraically.
- Sum and product of the roots of polynomial equations.
- Solutions of $a^x=b$ using logarithms
- Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.
- 2.7 Solutions of $g(x) \geq f(x)$.
- Graphical or algebraic methods, for simple polynomials up to degree 3.
- Use of technology for these and other functions.

Syllabus - Topic 3 - Core: Circular functions and trigonometry (22 hours)

The aims of this topic are to explore the circular functions, to introduce some important trigonometric identities and to solve triangles using trigonometry. On examination papers, radian measure should be assumed unless otherwise indicated, for example, by $x \mapsto \sin x°$.

- 3.1 The circle: radian measure of angles.
- Length of an arc; area of a sector.
- 3.2 Definition of $\cos\Theta$, $\sin\Theta$, $\tan\Theta$ in terms of the unit circle.
- Exact values of sin, cos and tan of 0, $\Pi/6$, $\Pi/4$, $\Pi/3$, $\Pi/2$ and their multiples.
- Definition of the reciprocal trigonometric ratios $\sec\Theta$, $\csc\Theta$ and $\cot\Theta$.
- Pythagorean identities: $\cos^2\Theta + \sin^2\Theta=1$;
- $1+\tan^2\Theta = \sec^2$; $1+\cot^2\Theta=\csc^2\Theta$.
- 3.3 Compound angle identities.
- Double angle identities.
- Not required:
- Proof of compound angle identities.
- 3.4 Composite functions of the form $f(x)= a \sin(b(x+c))+d$.
- Applications.
- 3.5 The inverse functions $x \mapsto \arcsin x$, $x \mapsto \arccos x$, $x \mapsto \arctan x$; their domains and ranges; their graphs.
- 3.6 Algebraic and graphical methods of solving trigonometric equations in a finite interval, including the use of trigonometric identities and factorization.
- Not required:
- The general solution of trigonometric equations.
- 3.7 The cosine rule
- The sine rule including the ambiguous case.
- Area of a triangle as 1/2 absinC.
- Applications.

Syllabus - Topic 4 - Core: Vectors (24 hours)

The aim of this topic is to introduce the use of vectors in two and three dimensions, and to facilitate solving problems involving points, lines and planes.

- 4.1 Concept of a vector.
- Representation of vectors using directed line segments.
- Unit vectors; base vectors i, j, k.
- Components of a vector: *equation not displayed
- Algebraic and geometric approaches to the following:
- • the sum and difference of two vectors;
- • the zero vector 0, the vector –v;
- • multiplication by a scalar, kv;
- • magnitude of a vector, |v|;
- • position vectors OA=a.
- AB= b-a
- 4.2 The definition of the scalar product of two vectors.
- Properties of the scalar product:
- $v \cdot w = w \cdot v$:
- $u \cdot (v+w)=u \cdot v+u \cdot w$;
- $(kv) \cdot w=k(v \cdot w)$;

- $v \cdot v = |v|^2$.
- The angle between two vectors.
- Perpendicular vectors; parallel vectors.
- 4.3 Vector equation of a line in two and three dimensions: $r = a + \lambda b$.
- Simple applications to kinematics.
- The angle between two lines.
- 4.4 Coincident, parallel, intersecting and skew lines; distinguishing between these cases.
- Points of intersection.
- 4.5 The definition of the vector product of two vectors.
- Properties of the vector product:
- $v \times w = -w \times v$;
- $u \times (v+w) = u \times v + u \times w$;
- $(kv) \times w = k(v \times w)$;
- $v \times v = 0$.
- Geometric interpretation of $|v \times w|$.
- 4.6 Vector equation of a plane $r = a + \lambda b + \mu c$.
- Use of normal vector to obtain the form $r \cdot n = a \cdot n$.
- Cartesian equation of a plane $ax+by+cz=d$.
- 4.7 Intersections of: a line with a plane; two planes; three planes.
- Angle between: a line and a plane; two planes.

Syllabus - Topic 5 - Core: Statistics and probability (36 hours)

The aim of this topic is to introduce basic concepts. It may be considered as three parts: manipulation and presentation of statistical data (5.1), the laws of probability (5.2–5.4), and random variables and their probability distributions (5.5–5.7). It is expected that most of the calculations required will be done on a GDC. The emphasis is on understanding and interpreting the results obtained. Statistical tables will no longer be allowed in examinations.

- 5.1 Concepts of population, sample, random sample and frequency distribution of discrete and continuous data.
- Grouped data: mid-interval values, interval width, upper and lower interval boundaries.
- Mean, variance, standard deviation.
- Not required:
- Estimation of mean and variance of a population from a sample.
- 5.2 Concepts of trial, outcome, equally likely outcomes, sample space (U) and event.
- The probability of an event A as $P(A) = (n(A))/(n(U))$.
- The complementary events A and A' (not A).
- Use of Venn diagrams, tree diagrams, counting principles and tables of outcomes to solve problems.
- 5.3 Combined events; the formula for $P(A \smile B)$.
- Mutually exclusive events.
- 5.4 Conditional probability; the definition $P(A|B) = (P(A \frown B)/P(B))$.
- Independent events; the definition $P(A|B)=P(A)=P(A|B')$.
- Use of Bayes’ theorem for a maximum of three events.
- 5.5 Concept of discrete and continuous random variables and their probability distributions.
- Definition and use of probability density functions.
- Expected value (mean), mode, median, variance and standard deviation.
- Applications.
- 5.6 Binomial distribution, its mean and variance.
- Poisson distribution, its mean and variance.
- Not required:
- Formal proof of means and variances.
- 5.7 Normal distribution.
- Properties of the normal distribution.
- Standardization of normal variables.

Syllabus - Topic 6 - Core: Calculus (48 hours)

The aim of this topic is to introduce students to the basic concepts and techniques of differential and integral calculus and their application.

- 6.1 Informal ideas of limit, continuity and convergence.
- Definition of derivative from first principles. *Equation not displayed
- The derivative interpreted as a gradient function and as a rate of change.
- Finding equations of tangents and normals.
- Identifying increasing and decreasing functions.
- The second derivative.
- Higher derivatives.
- 6.2 Derivatives of x^n, sin x, cos x, tan x, e^x and ln x.
- Differentiation of sums and multiples of functions.
- The product and quotient rules.
- The chain rule for composite functions.
- Related rates of change.
- Implicit differentiation.

- Derivatives of sec x, csc x, cot x, a^x, $\log_a x$, arcsin x, arccos x and arctan x.
- 6.3 Local maximum and minimum values.
- Optimization problems.
- Points of inflexion with zero and non-zero gradients.
- Graphical behaviour of functions, including the relationship between the graphs of f, f', f''.
- Not required:
- Points of inflexion, where $f''(x)$ is not defined, for example, $y = x^{1/3}$ at (0,0).
- 6.4 Indefinite integration as anti-differentiation.
- Indefinite integral of x^n, sin x, cos x and e^x.
- Other indefinite integrals using the results from 6.2.
- The composites of any of these with a linear function.
- 6.5 Anti-differentiation with a boundary condition to determine the constant of integration.
- Definite integrals.
- Area of the region enclosed by a curve and the x-axis or y-axis in a given interval; areas of regions enclosed by curves.
- Volumes of revolution about the x-axis or y-axis.
- 6.6 Kinematic problems involving displacement s, velocity v and acceleration a.
- Total distance travelled.
- 6.7 Integration by substitution
- Integration by parts.

Syllabus - Topic 7 - Option: Statistics and probability (48 hours)

The aims of this option are to allow students the opportunity to approach statistics in a practical way; to demonstrate a good level of statistical understanding; and to understand which situations apply and to interpret the given results. It is expected that GDCs will be used throughout this option, and that the minimum requirement of a GDC will be to find probability distribution function (pdf), cumulative distribution function (cdf), inverse cumulative distribution function, p-values and test statistics, including calculations for the following distributions: binomial, Poisson, normal and t. Students are expected to set up the problem mathematically and then read the answers from the GDC, indicating this within their written answers. Calculator-specific or brand-specific language should not be used within these explanations.

- 7.1 Cumulative distribution functions for both discrete and continuous distributions.
- Geometric distribution.
- Negative binomial distribution.
- Probability generating functions for discrete random variables.
- Using probability generating functions to find mean, variance and the distribution of the sum of n independent random variables.
- 7.2 Linear transformation of a single random variable.
- Mean of linear combinations of n random variables.
- Variance of linear combinations of n independent random variables.
- Expectation of the product of independent random variables.
- 7.3 Unbiased estimators and estimates.
- Comparison of unbiased estimators based on variances.
- X-Bar as an unbiased estimator for μ.
- S^2 as an unbiased estimator for σ^2.
- 7.4 A linear combination of independent normal random variables is normally distributed. In particular, $\bar{X} \backsim N(\mu, \sigma^2) \Rightarrow \bar{X} \backsim N(\mu, (\sigma^2/n))$.
- The central limit theorem.
- 7.5 Confidence intervals for the mean of a normal population.
- 7.6 Null and alternative hypotheses, Null and alternative hypotheses, H_0 and H_1 .
- Significance level.
- Critical regions, critical values, p-values, one-tailed and two-tailed tests.
- Type I and II errors, including calculations of their probabilities.
- Testing hypotheses for the mean of a normal population.
- 7.7 Introduction to bivariate distributions.
- Covariance and (population) product moment correlation coefficient p.
- Proof that p= 0 in the case of independence and ±1 in the case of a linear relationship between X and Y.
- Definition of the (sample) product moment correlation coefficient R in terms of n paired observations on X and Y. Its application to the estimation of p.
- Informal interpretation of r, the observed value of R. Scatter diagrams.
- The following topics are based on the assumption of bivariate normality.
- Use of the t-statistic to test the null hypothesis p = 0.
- Knowledge of the facts that the regression of X on Y(E(X)|Y=y) and Y on X (E(Y)|X=x) are linear.
- Least-squares estimates of these regression lines (proof not required).
- The use of these regression lines to predict the value of one of the variables given the value of the other.

Syllabus - Topic 8 - Option: Sets, relations and groups (48 hours)

The aims of this option are to provide the opportunity to study some important mathematical concepts, and introduce the principles of proof through abstract algebra.

- 8.1 Finite and infinite sets. Subsets.
- Operations on sets: union; intersection; complement; set difference; symmetric difference.
- De Morgan's laws: distributive, associative and commutative laws (for union and intersection).
- 8.2 Ordered pairs: the Cartesian product of two sets.
- Relations: equivalence relations; equivalence classes.
- 8.3 Functions: injections; surjections; bijections.

- Composition of functions and inverse functions.
- 8.4 Binary operations.
- Operation tables (Cayley tables).
- 8.5 Binary operations: associative, distributive and commutative properties.
- 8.6 The identity element e.
- The inverse a^{-1} of an element a.
- Proof that left-cancellation and right-cancellation by an element a hold, provided that a has an inverse.
- Proofs of the uniqueness of the identity and inverse elements.
- 8.7 The definition of a group {G ,*}.
- The operation table of a group is a Latin square, but the converse is false.
- Abelian groups.
- 8.8 Examples of groups:
- • $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ under addition;
- • integers under addition modulo n;
- • non-zero integers under multiplication, modulo p, where p is prime;
- symmetries of plane figures, including equilateral triangles and rectangles;
- invertible functions under composition of functions.
- 8.9 The order of a group.
- The order of a group element.
- Cyclic groups.
- Generators.
- Proof that all cyclic groups are Abelian.
- 8.10 Permutations under composition of permutations.
- Cycle notation for permutations.
- Result that every permutation can be written as a composition of disjoint cycles.
- The order of a combination of cycles.
- 8.11 Subgroups, proper subgroups.
- Use and proof of subgroup tests.
- Definition and examples of left and right cosets of a subgroup of a group.
- Lagrange's theorem.
- Use and proof of the result that the order of a finite group is divisible by the order of any element. (Corollary to Lagrange's theorem.)
- 8.12 Definition of a group homomorphism.
- Definition of the kernel of a homomorphism. Proof that the kernel and range of a homomorphism are subgroups.
- Proof of homomorphism properties for identities and inverses.
- Isomorphism of groups.
- The order of an element is unchanged by an isomorphism.

Syllabus - Topic 9 - Option: Calculus (48 hours)

The aims of this option are to introduce limit theorems and convergence of series, and to use calculus results to solve differential equations.

- 9.1 Infinite sequences of real numbers and their convergence or divergence.
- 9.2 Convergence of infinite series.
- Tests for convergence: comparison test; limit comparison test; ratio test; integral test.
- The p-series, $\sum(1/n^p)$.
- Series that converge absolutely. Series that converge conditionally. Alternating, series.
- Power series: radius of convergence and interval of convergence. Determination of the radius of convergence by the ratio test.
- 9.3 Continuity and differentiability of a function at a point.
- Continuous functions and differentiable functions.
- 9.4 The integral as a limit of a sum; lower and upper Riemann sums.
- Fundamental theorem of calculus.
- Improper integrals of the type *equation not displayed
- 9.5 First-order differential equations.
- Geometric interpretation using slope fields, including identification of isoclines.
- Numerical solution of Dy/dx= f(x, y) using Euler's method.
- Variables separable.
- Homogeneous differential equation *equation not displayed using the substitution y = vx.
- Solution of y′ + P(x)y = Q(x), using the integrating factor.
- 9.6 Rolle's theorem.
- Mean value theorem.
- Taylor polynomials; the Lagrange form of the error term.
- Maclaurin series for e^x , sin x , cos x , ln(1+x) , $(1+x)^P$, p $\varepsilon\mathbb{Q}$.
- Use of substitution, products, integration and differentiation to obtain other series.
- Taylor series developed from differential equations.
- 9.7 The evaluation of limits of the form *equation not displayed
- Using l'Hôpital's rule or the Taylor series.

Syllabus - Topic 10 - Option: Discrete mathematics (48 hours)

The aim of this option is to provide the opportunity for students to engage in logical reasoning, algorithmic thinking and applications.

- 10.1 Strong induction.
- Pigeon-hole principle.
- 10.2 $a|b \Rightarrow b = na$ for some $N \in \mathbb{Z}$
- The theorem $a|b$ and $a|c \Rightarrow a \mid (bx \pm cy)$ where $x,y \in \mathbb{Z}$.
- Division and Euclidean algorithms.
- The greatest common divisor, gcd(a, b) , and the least common multiple, lcm(a, b) , of integers a and b.
- Prime numbers; relatively prime numbers and the fundamental theorem of arithmetic.
- 10.3 Linear Diophantine equations $ax + by = c$.
- 10.4 Modular arithmetic.
- The solution of linear congruences.
- Solution of simultaneous linear congruences. (Chinese remainder theorem).
- 10.5 Representation of integers in different bases.
- 10.6 Fermat's little theorem.
- 10.7 Graphs, vertices, edges, faces. Adjacent vertices, adjacent edges.
- Degree of a vertex, degree sequence.
- Handshaking lemma.
- Simple graphs; connected graphs; complete graphs; bipartite graphs; planar graphs; trees; weighted graphs, including tabular representation.
- Subgraphs; complements of graphs.
- Euler's relation: $v - e + f = 2$; theorems for planar graphs including $e \leq 3v-6$, $e \leq 2v-4$, leading to the results that K_5 and $K_{3,3}$ are not planar.
- 10.8 Walks, trails, paths, circuits, cycles.
- Eulerian trails and circuits.
- Hamiltonian paths and cycles.
- 10.9 Graph algorithms: Kruskal's; Dijkstra's.
- 10.10 Chinese postman problem.
- Not required:
- Graphs with more than four vertices of odd degree.
- Travelling salesman problem.
- Nearest-neighbour algorithm for determining an upper bound.
- Deleted vertex algorithm for determining a lower bound.
- 10.11 Recurrence relations. Initial conditions, recursive definition of a sequence.
- Solution of first- and second-degree linear homogeneous recurrence relations with constant coefficients.
- The first-degree linear recurrence relation $u_n = au_{n-1} + b$.
- Modelling with recurrence relations.

Internal assessment criteria

Internal assessment criteria

- Criterion A: Communication
- Criterion B: Mathematical presentation
- Criterion C: Personal engagement
- Criterion D: Reflection
- Criterion E: Use of mathematics